

Patterns of primes in arithmetic progressions

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1 Introduction

In their ground-breaking work Green and Tao [GT 2008] proved the existence of infinitely many k -term arithmetic progressions in the sequence of primes for every integer $k > 0$. I showed a conditional strengthening of it [Pin 2010] according to which if the primes have a distribution level $\vartheta > 1/2$ (for the definition of the distribution level see (1.1) below), then there exists a constant $C(\vartheta)$ such that we have a positive even $d \leq C(\vartheta)$ with the property that $0 < d \leq C(\vartheta)$ and for every k there exist infinitely many arithmetic progressions $\{p_i^*\}_{i=1}^k$ of length k with $p_i^* \in \mathcal{P}$ (\mathcal{P} denotes the set of primes) such that $p_i^* + d$ is a prime too, in particular, the prime following p_i^* . After the proof of Zhang [Zhang 2014], proving the unconditional existence of infinitely many bounded gaps between primes (this was proved earlier in our work [GPY 2009] under the condition that primes have a distribution level $\vartheta > 1/2$) I showed this without any unproved hypotheses [Pin 2015].

We say that θ is a distribution level of the primes if

$$(1.1) \quad \sum_{q \leq x^\theta} \max_{\substack{a \\ (a,q)=1}} \left| \pi(x, q, a) - \frac{\pi(x)}{\varphi(q)} \right| \ll_A \frac{x}{(\log x)^A}$$

holds for any $A > 0$ where the \ll symbol of Vinogradov means that $f(x) = O(g(x))$ is abbreviated by $f(x) \ll g(x)$.

In his recent work James Maynard [May 2015] gave a simpler and more efficient proof of Zhang's theorem. In particular he gave an unconditional proof of a weaker version of Dickson's conjecture [Dic 1904] which we abbreviate as Conjecture DHL since Hardy and Littlewood formulated a stronger quantitative version of it twenty years later [HL 1923].

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Conjecture DHL (Prime k -tuples Conjecture). *Let $\mathcal{H} = \{h_1, \dots, h_k\}$ be admissible, which means that for every prime p there exists an integer a_p such that for any i $a_p \not\equiv h_i \pmod{p}$. Then there are infinitely many integers n such that all of $n + h_1, \dots, n + h_k$ are primes.*

The weaker version showed by Maynard (and simultaneously and independently by T. Tao (unpublished)) was that Conjecture DHL (k, k_0) (formulated below) holds for $k \gg k_0^2 e^{4k_0}$.

Conjecture DHL (k, k_0) . *If \mathcal{H} is admissible of size k , then there are infinitely many integers n such that $\{n + h_i\}_{i=1}^k$ contains at least k_0 primes.*

A brief argument, given by Maynard [May 2015] (see Theorem 1.2 of his work) shows that if there exists a $C(k_0)$ such that DHL (k, k_0) holds for $k \geq C(k_0)$, then a positive proportion of all admissible m -tuples satisfy the prime m -tuple conjecture for every m (for the exact formulation see Theorem 1.2 of [May 2015]).

The purpose of the present work is to show a common generalization of the result of Maynard (and Tao) and that of Green–Tao.

Theorem 1. *Let $m > 0$ and $\mathcal{A} = \{a_1, \dots, a_n\}$ be a set of r distinct integers with r sufficiently large depending on m . Let $N(\mathcal{A})$ denote the number of integer m -tuples $\{h_1, \dots, h_m\} \subseteq \mathcal{A}$ such that there exist for every ℓ infinitely many ℓ -term arithmetic progressions of primes $\{p_i^*\}_{i=1}^\ell$ where $p_i^* + h_j$ is also prime for each pair i, j . Then*

$$(1.2) \quad N(\mathcal{A}) \gg_m \#\{(h_1, \dots, h_m) \in \mathcal{A}\} \gg_m |\mathcal{A}|^m = r^m.$$

This is an unconditional generalization of the result in [Pin 2010].

2 Preparation. First part of the proof of Theorem 2

The arguments in the last three paragraphs of Section 4 of [May 2015] can be applied here practically without any change and so, similarly to Theorems 1.1 and 1.2 of [May 2015], our Theorem 1 will also follow in essentially the same way from (the weaker)

Theorem 2. *Let m be a positive integer, $\mathcal{H} = \{h_1, \dots, h_k\}$ be an admissible set of k distinct non-negative integers $h_i \leq H$, $k = \lceil Cm^2 e^{4m} \rceil$ with a*

sufficiently large absolute constant C . Then there exists an m -element subset

$$(2.1) \quad \{h'_1, h'_2, \dots, h'_m\} \subseteq \mathcal{H}$$

such that for every positive integer ℓ we have infinitely many ℓ -element non-trivial arithmetic progressions of primes p_i^* such that $p_i^* + h'_j \in \mathcal{P}$ for $1 \leq i \leq \ell$, $1 \leq j \leq m$, further $p_i^* + h'_j$ is always the j -th prime following p_i^* .

Remark.

- (i) For $\ell = m = 1$ this is Zhang's theorem,
- (ii) for $\ell = 1$, m arbitrary this is the Maynard–Tao theorem,
- (iii) for $m = 0$, ℓ arbitrary this is the Green–Tao theorem,
- (iv) for $m = 1$, ℓ arbitrary this was proved under the condition that primes have a distribution level $\theta > 1/2$ in [Pin 2010], unconditionally (using Zhang's method) in [Pin 2015].

In order to show our Theorem 2 we will follow the scheme of [May 2015]. We therefore emphasize just a few notations here, but we will use everywhere Maynard's notation throughout our work. Similarly to his work, k will be a fixed integer, $\mathcal{H} = \{h_1, \dots, h_k\} \subseteq [0, H]$ a fixed admissible set. Any constants implied by the \ll and O notations may depend on k and H . N will denote a large integer and asymptotics will be understood as $N \rightarrow \infty$. Most variables will be natural numbers, p (with or without subscripts) will denote always primes, $[a, b]$ the least common multiple of $[a, b]$ (however, sometimes the closed interval $[a, b]$). We will weight the integers with a non-negative weight w_n which will be zero unless n lies in a fixed residue class $\nu_0 \pmod{W}$ where $W = \prod_{p \leq D_0} p$. D_0 tends in [May 2015] slowly to infinity with N . His choice is actually $D_0 = \log \log \log N$. However, it is sufficient to choose

$$(2.2) \quad D_0 = C^*(k),$$

with a sufficiently large constant $C^*(k)$, depending on k .

The proof runs similarly in this case as well just we lose the asymptotics then, but the dependence on D_0 is explicitly given in [May 2015]. The weights w_n are defined in (2.4) of [May 2015] as

$$(2.3) \quad w_n = \left(\sum_{d_i | n + h_i \forall i} \lambda_{d_1, \dots, d_k} \right)^2.$$

The choice of $\lambda_{d_1, \dots, d_k}$ will be through the choice of other parameters y_{r_1, \dots, r_k} by the aid of the identity

$$(2.4) \quad \lambda_{d_1, \dots, d_k} = \left(\prod_{i=1}^k \mu(d_i) d_i \right) \sum_{\substack{r_1, \dots, r_k \\ d_i | r_i \forall i \\ (r_i, W)=1}} \frac{\mu \left(\prod_{i=1}^k r_i \right)^2}{\prod_{i=1}^k \varphi(r_i)} y_{r_1, \dots, r_k}$$

whenever $\left(\prod_{i=1}^k d_i, W \right) = 1$ and $\lambda_{d_1, \dots, d_k} = 0$ otherwise. Here y_{r_1, \dots, r_k} will be defined by the aid of a piecewise differentiable function F , the distribution $\theta > 0$ of the primes, with $R = N^{\theta/2-\varepsilon}$ as

$$(2.5) \quad y_{r_1, \dots, r_k} = F \left(\frac{\log r_1}{\log R}, \dots, \frac{\log r_k}{\log R} \right)$$

where F will be real valued, supported on

$$(2.6) \quad R_k = \left\{ (x_1, \dots, x_k) \in [0, 1]^k : \sum_{i=1}^k x_i \leq 1 \right\}.$$

All this is in complete agreement with the notation of Proposition 1 and (6.3) of [May 2015].

Our proof will also make use of the main pillars of Maynard's proof, his Propositions 1–3, which we quote now with the above notations as

Proposition 1'. *With the above notation let*

$$(2.7) \quad S_1 := \sum_{\substack{n \\ N \leq n < 2N \\ n \equiv \nu_0 \pmod{W}}} w_n, \quad S_2 := \sum_{\substack{n \\ N \leq n < 2N \\ n \equiv \nu_0 \pmod{W}}} \left(w_n \sum_{i=1}^k \chi_{\mathcal{P}}(n + h_i) \right),$$

where $\chi_{\mathcal{P}}(n)$ denotes the characteristic function of the primes. Then we have as $N \rightarrow \infty$

$$(2.8) \quad S_1 = \frac{\left(1 + O \left(\frac{1}{D_0} \right) \right) \varphi(W)^k N (\log R)^k}{W^{k+1}} I_k(F),$$

$$(2.9) \quad S_2 = \frac{\left(1 + O \left(\frac{1}{D_0} \right) \right) \varphi(W)^k N (\log R)^{k+1}}{W^{k+1}} \sum_{j=1}^k J_k^{(j)}(F),$$

provided $I_k(F) \neq 0$ and $J_k^{(j)}(F) \neq 0$ for each j , where

$$(2.10) \quad I_k(F) = \int_0^1 \dots \int_0^1 F(t_1, \dots, t_k)^2 dt_1 \dots dt_k,$$

$$(2.11) \quad I_k^{(j)}(F) = \int_0^1 \dots \int_0^1 \left(\int_0^1 F(t_1, \dots, t_k) dt_j \right)^2 dt_1 \dots dt_{j-1} dt_{j+1} \dots dt_k.$$

Proposition 2'. Let \mathcal{S}_k denote the set of piecewise differentiable functions with the earlier given properties, including $I_k(F) \neq 0$ and $J_k^{(j)}(F) \neq 0$ for $1 \leq j \leq k$. Let

$$(2.12) \quad M_k = \sup \frac{\sum_{j=1}^k J_k^{(j)}(F)}{I_k(F)}, \quad r_k = \left\lceil \frac{\theta M_k}{2} \right\rceil$$

and let \mathcal{H} be a fixed admissible sequence $\mathcal{H} = \{h_1, \dots, h_k\}$ of size k . Then there are infinitely many integers n such that at least r_k of the $n + h_i$ ($1 \leq i \leq k$) are simultaneously primes.

Proposition 3'. $M_{105} > 4$ and $M_k > \log k - 2 \log \log k - 2$ for $k > k_0$.

Remark. In the proof Maynard will use for every k an explicitly given function $F = F_k$ satisfying the above inequality. Therefore the additional dependence on F will be actually a dependence on k .

The main idea (beyond the original proof of Maynard–Tao) is that in the weighted sum S_1 in (2.7) all those weights w_n for numbers $n \in [N, 2N]$ are in total negligible for which any of the $n + h_i$ terms ($1 \leq i \leq k$) has a small prime factor p (i.e. with a sufficiently small $c_1(k)$ depending on k , $p \mid n + h_i$, $p < n^{c_1(k)}$).

To make it more precise let $c_1(k)$ be a sufficiently small fixed constant (to be determined later and fixed for the rest of the work). Let $P^-(n)$ be the smallest prime factor of n . Then we have

Lemma 1. *We have*

$$(2.13) \quad S_1^- = \sum_{\substack{N \leq n < 2N \\ n \equiv \nu_0 \pmod{W}}} w_n \ll_{k,H} \frac{c_1(k) \log N}{\log R} S_1. \\ P^- \left(\prod_{i=1}^k (n + h_i) \right) < n^{c_1(k)}$$

Since $R = N^{\frac{\theta}{2}-\varepsilon}$, S_1^-/S_1 will be arbitrarily small if $c_1(k)$ is chosen sufficiently small. The proof of Lemma 1 will be postponed to Section 3. This means that during the whole proof we can neglect those numbers n for which $P^-\left(\prod_{i=1}^k(n+h_i)\right) < n^{c_1(k)}$ and it is sufficient to deal with numbers n with $n+h_i$ being almost primes for each $i = 1, 2, \dots, k$ (by which we mean that $n+h_i$ has only prime factors at least $n^{c_1(k)}$). A trivial consequence of this fact is that for such numbers n $\prod_{i=1}^k(n+h_i)$ has a bounded number of prime factors. Consequently we have for these numbers n by (5.9) and (6.3)

$$(2.14) \quad w_n \ll_{c_1(k),k} \lambda_{\max}^2 \ll_{c_1(k),k} y_{\max}^2 (\log R)^{2k} \ll_{c_1(k),k,F} (\log R)^{2k} \ll (\log R)^{2k}$$

with the convention that the constants implied by the \ll and O constants can depend on k and both $c_1(k)$ and $F = F_k$ will only depend on k .

The essence of Maynard's proof is that (see (4.1)–(4.4) of [May 2015])

$$(2.15) \quad S_2 > \left(\left(\frac{\theta}{2} - \varepsilon \right) (M_k - \varepsilon) + O\left(\frac{1}{D_0} \right) \right) S_1$$

which directly implies the existence of infinitely many values n such that there are at least

$$(2.16) \quad r_k = \left\lceil \frac{\theta M_k}{2} \right\rceil$$

primes among $n+h_i$ ($1 \leq i \leq k$).

Let us denote, in analogy with (2.7)

$$(2.17) \quad S_1^+ := \sum_{\substack{N \leq n < 2N \\ n \equiv \nu_0 \pmod{W} \\ P^-\left(\prod_{i=1}^k(n+h_i)\right) \geq n^{c_1(k)}} w_n, \quad S_2^+ := \sum_{\substack{N \leq n < 2N \\ n \equiv \nu_0 \pmod{W} \\ P^-\left(\prod_{i=1}^k(n+h_i)\right) \geq n^{c_1(k)}} w_n \left(\sum_{i=1}^k \chi_{\mathcal{P}}(n+h_i) \right).$$

Then Lemma 1, i.e. (2.13) implies together with (2.15) that (if $c_1(k)$ and ε are chosen sufficiently small, D_0 sufficiently large, then)

$$(2.18) \quad S_2^+ > \left(\left(\frac{\theta}{2} - \varepsilon \right) (M_k - \varepsilon) + O(c_1(k)) + O\left(\frac{1}{D_0} \right) + o(1) \right) S_1,$$

which implies the existence of a large number of n values in $[N, 2N)$, $n \equiv \nu_0 \pmod{W}$ with at least r_k primes among them and additionally almost primes with $P^-(n + h_i) > n^{c_1(k)}$ in all other components $i \in [1, k]$.

Together with (2.14) this implies
(2.19)

$$S_1^* := \sum_{\substack{N \leq n < 2N \\ n \equiv \nu_0 \pmod{W} \\ P^-\left(\prod_{i=1}^k (n+h_i)\right) > n^{c_1(k)} \\ \#\{i; n+h_i \in \mathcal{P}\} \geq r_k}} 1 \gg \frac{S_1}{(\log R)^{2k}} = \frac{\left(1 + O\left(\frac{1}{D_0}\right)\right) \varphi(W)^k N I_k(F)}{W^{k+1} (\log R)^k}.$$

Since $D_0 = C^*(k)$ we have $\varphi(W)^k / W^{k+1} \geq C'(k)$. Thus a positive proportion (depending on k) of the integers $n \in [N, 2N)$ with $n \equiv \nu_0 \pmod{W}$ and $P^-\left(\prod_{i=1}^k (n + h_i)\right) > n^{c_1(k)}$ contain at least r_k primes among $n + h_i$ ($1 \leq i \leq k$). This follows from (2.19) and

$$(2.20) \quad \sum_{\substack{N \leq n < 2N, n \equiv \nu_0 \pmod{W} \\ P^-\left(\prod_{i=1}^k (n+h_i)\right) > n^{c_1(k)}}} 1 \ll \frac{N}{\log^k N}$$

where the implied constant in the \ll symbol depends only on k , H and $c_1(k)$, therefore only on k , finally. (2.20) is a consequence of Selberg's sieve (see, for example, Theorem 5.1 of [HR 1974] or Theorem 2 in § 2.2.2 of [Gre 2001]).

If Lemma 1 will be proved (see Section 3) then Theorem 2 will follow from Theorem 5 of [Pin 2010] which we quote here as

Main Lemma. *Let k be an arbitrary positive integer and $\mathcal{H} = \{h_1, \dots, h_k\}$ be an admissible k -tuple. If the set $\mathcal{N}(\mathcal{H})$ satisfies with constants $c_1(k)$, $c_2(k)$*

$$(2.21) \quad \mathcal{N}(\mathcal{H}) \subseteq \left\{ n; P^-\left(\prod_{i=1}^k (n + h_i)\right) \geq n^{c_1(k)} \right\}$$

and

$$(2.22) \quad \#\{n \leq X, n \in \mathcal{N}(\mathcal{H})\} \geq \frac{c_2(k)X}{\log^k X}$$

for $X > X_0$, then $\mathcal{N}(\mathcal{H})$ contains ℓ -term arithmetic progressions for every ℓ .

In order to see that the extra condition that the given prime pattern occurs also for consecutive primes we have to work in the following way. For any given $\mathcal{H} = \{h_1, \dots, h_k\}$ with $k = \lceil Cm^2 \log m \rceil$ we choose an m -element subset $\mathcal{H}' = \{h'_1, \dots, h'_m\} \subseteq \mathcal{H}$ with minimal diameter $h'_m - h'_1$ such that with some constants $c'_1(k), c'_2(k) > 0$ the relations (2.21)–(2.22), more exactly (2.23)

$$\# \left\{ n \leq X; P^- \left(\prod_{i=1}^k (n + h_i) \right) \geq n^{c'_1(k)}, n + h'_i \in \mathcal{P} \ (1 \leq i \leq m) \right\} \geq \frac{c'_2(k)X}{\log^k X}$$

should hold for $X > X_0$.

By the condition that \mathcal{H}' has minimal diameter we can delete from our set $\mathcal{N}(\mathcal{H})$ those n 's for which there exists any $h_i \in \mathcal{H} \setminus \mathcal{H}'$, $h'_1 < h_i < h'_m$ such that beyond (2.23) also $n + h_i \in \mathcal{P}$ would hold.

On the other hand we can also neglect those $n \in \mathcal{N}(\mathcal{H})$ for which with a given $h \in [1, H]$, $h \notin \mathcal{H}_k$ we would have additionally $n + h \in \mathcal{P}$ since the total number of such $h \in [1, H]$ is by (2.20) at most

$$(2.24) \quad O_k \left(\frac{NH}{\log^{k+1} N} \right) = o \left(\frac{N}{\log^k N} \right)$$

since our original H in Theorem 2 was fixed.

We note that the above way of specifying the m -element sets \mathcal{H}'_m for which we have arbitrarily long (finite) arithmetic progressions of n 's such that $n + h'_i$ ($1 \leq i \leq m$) would be a given bounded pattern of *consecutive* primes does not change the validity of the argument of Maynard (see Theorem 1.2 of [May 2015]) which shows that the above is true for a positive proportion of all m -element sets (the proportion depends on m).

3 Proof of Lemma 1. End of the proof of Theorem 2

The proof of Lemma 1 will be a trivial consequence of the following

Lemma 2. *The following relation holds for any prime $D_0 < p < N^{c_1}$ and all $i \in [1, \dots, k]$:*

$$(3.1) \quad S_{1,p}^* := \sum_{\substack{N \leq n < 2N \\ n \equiv \nu_0 \pmod{W} \\ p | n + h_i}} w_n \ll_{F,H,k} \frac{\log p}{p \log R} \sum_{\substack{N \leq n < 2N \\ n \equiv \nu_0 \pmod{W}}} w_n = \frac{\log p}{p \log R} S_1.$$

Proof. It is clear that it is enough to show this for $i = 1$, for example. During the proof we will use the analogue of Lemma 6 of [GGPY 2010] for the special case $k = 1$, $\delta = p \in \mathcal{P}$ and for squarefree n with

$$(3.2) \quad f(n) = n \quad f_1(n) = \mu * f(n) = \prod_{p|n} (p-1) = \varphi(n)$$

which is as follows:

$$(3.3) \quad T_p := \sum_{d,e} \frac{\lambda_d \lambda_e}{[d, e, p]/p} = \sum_{\substack{r \\ p+r}} \frac{\mu^2(r)}{\varphi(r)} (y_r - y_{rp})^2.$$

This form appears as the last displayed equation on page 85 of Selberg [Sel 1991] or equation (1.9) on page 287 of Greaves [Gre 2001]. We note the general starting condition that similarly to [May 2015] the numbers W , $[d_1, e_1], \dots, [d_k, e_k]$ will be always coprime to each other.

Writing $n + h_1 = pm$ we see that we have for any $\varepsilon > 0$ and denoting \sum^* for the conditions $n \in [N, 2N)$, $n \equiv \nu_0 \pmod{W}$; $d_i, e_i \pmod{n + h_i}$ ($2 \leq i \leq k$)

$$(3.4) \quad S_{1,p} = \sum_1 + \sum_2 + O(R^{2+\varepsilon})$$

where

$$(3.5) \quad \sum_1 = \sum_{\substack{p \nmid [d_1, e_1] \\ d_1 | m, e_1 | m}}^* \lambda_{d_1, \dots, d_k} \lambda_{e_1, \dots, e_k},$$

$$(3.6) \quad \sum_2 = \sum_{\substack{p \mid [d_1, e_1] \\ d_1 | pm, e_1 | pm}}^* \lambda_{d_1, \dots, d_k} \lambda_{e_1, \dots, e_k}.$$

Distinguishing further in \sum_2 according to $p^2 \mid d_1 e_1$ or not we obtain from (3.5) and (3.6) for any $\varepsilon > 0$

$$(3.7) \quad \sum_1 = \frac{N}{pW} \sum_{p \nmid [d_1, e_1]}^* \frac{\lambda_{d_1, \dots, d_k} \lambda_{e_1, \dots, e_k}}{\prod_{i=1}^k [d_i, e_i]} + O(R^{2+\varepsilon})$$

and

$$(3.8) \quad \sum_2 = \frac{N}{pW} \left\{ \left(\sum_{d_1=pd'_1, p \nmid e_1}^* \frac{\lambda_{pd'_1, \dots, d_k} \lambda_{e_1, \dots, e_k}}{\prod_{i=2}^k [d_i, e_i]} + \sum_{e_1=pe'_1, p \nmid d_1}^* \frac{\lambda_{d_1, \dots, d_k} \lambda_{pe'_1, \dots, e_k}}{\prod_{i=2}^k [d_i, e_i]} \right) \right. \\ \left. + \sum_{d_1=pd'_1, e_1=pe'_1}^* \frac{\lambda_{pd'_1, \dots, d_k} \lambda_{e'_1 p, \dots, e_k}}{\prod_{i=1}^k [d_i, e_i]} \right\} + O(R^{2+\varepsilon}).$$

Consequently we have

$$(3.9) \quad S_{1,p} = \frac{N}{pW} \sum^* \frac{\lambda_{d_1, \dots, d_k} \lambda_{e_1, \dots, e_k}}{\frac{[d_1, e_1, p]}{p} \prod_{i=2}^k [d_i, e_i]} + O(R^{2+\varepsilon}).$$

Let us denote the sum in (3.9) analogously to (3.3) by $T_{p,1}$. Then, similarly to (3.3) we obtain using additionally the argument of Section 5 of [May 2015]

$$(3.10) \quad T_{p,1} = \sum_{u_1, \dots, u_k} \frac{\prod_{i=1}^k \mu^2(u_i)}{\prod_{i=1}^k \varphi(u_i)} (y_{u_i, \dots, u_k} - y_{u_1 p, u_2, \dots, u_k})^2.$$

However by the choice (6.3) of [May 2015] we have

$$(3.11) \quad (y_{u_1, \dots, u_k} - y_{u_1 p, u_2, \dots, u_k})^2 = F\left(\frac{\log u_1}{\log R}, \dots, \frac{\log u_k}{\log R}\right)^2 - F\left(\frac{\log u_1 + \log p}{\log R}, \dots, \frac{\log u_k}{\log R}\right)^2 \\ \ll_F \frac{\log p}{\log R} \ll \frac{\log p}{\log R},$$

since F depends only on k , and hence the constant implied by the \ll symbol may depend on k . Hence we have by Proposition (4.1) of [May 2015]

$$(3.12) \quad T_{p,1} \ll \frac{N}{W} \cdot \frac{\log p}{p \log R} \sum_{(u_1, W)=1} \frac{\prod_{i=1}^k (\mu^2(u_i))}{\prod_{i=1}^k \varphi(u_i)} \ll \frac{\log p}{p \log R} \cdot S_1$$

which proves Lemma 2 and thereby Lemma 1 and Theorem 2. \square

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